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Withdrawal of fluid through a line sink beneath a free surface above a sloping boundary

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Abstract. A series truncation method is used to compute the flow into a line sink from a region of fluid with a free surface and a sloping wall beneath the sink. The method admits a well-known exact solution for a particular value of the slope. Solutions with a cusp above the sink, and with a stagnation point above the sink are computed for all values of the slope, and compared with results at both ends of the range, i.e. with results for both a vertical wall and a horizontal bottom, with good agreement.

1. Introduction

In this paper the problem of withdrawal of water though a line sink situated beneath a free surface is considered. The free surface can be thought of as an air-water interface, and the line sink as a slot through which the water is being withdrawn. The water body is assumed to be infinitely deep with semi-infinite horizontal extent. The line sink is situated at a corner in the boundary, which then slopes away with angle $\pi\gamma$ (see Fig. 1). Note that with this geometry, the problem is essentially two dimensional. One reason for choosing this geometry is the existence of an exact solution to the full non-linear problem when $\gamma = 1/3$ [1–3].

Apart from the direct application, this problem has relevance to the process of withdrawal of water from layers of fluid of different density, as in a reservoir, cooling pond or solar pond, where the action of the weather and inflows and withdrawals can sometimes lead to the formation of homogeneous layers of different density [4]. If the free surface is replaced by an interface of infinitessimal thickness to a fluid of lesser density, and it is assumed that the lighter layer is stagnant, then it is possible to replace the gravity g by an effective gravity $g' = (\Delta\rho/\rho)g$, where $\Delta\rho$ is the density difference between the layers, and ρ is some reference density. The condition of constant pressure along the interface reduces to the same condition of constant pressure along a free surface acted upon by this modified gravity g'. The

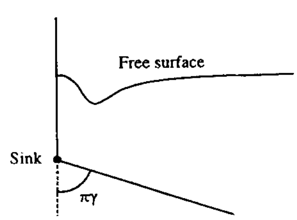

Fig. 1. This figure shows the geometry being considered in this paper. A line sink is withdrawing fluid from a water body beneath a free surface above a sloping boundary.

assumptions necessary are of arguable validity, since they neglect the effects of viscosity at the interface, which would undoubtedly induce some motion in the upper fluid. However, the numerical solutions follow the qualitative behaviour of the experiments [5–9] quite well.

For this problem with the geometry described above, there are two important parameters governing the flow. The first is the Froude number F, which gives an indication of the strength of the sink compared to the weight of fluid above it, and the second is γ, the angle at which the boundary slopes away from the sink.

The Froude number is usually defined as $F = q/(gH^3)^{1/2}$, where q is the flux per unit length into the sink, g is the acceleration due to gravity, and H is the depth of the sink beneath the undisturbed free surface. In this case the flux into the line sink is given by $m(1-\gamma)$, where m is the strength of the sink, since the fluid can only enter the sink through an arc of $\pi(1-\gamma)$. Thus, the definition of the Froude number used here will be

$$F = \frac{m(1-\gamma)}{(gH^3)^{1/2}} . \tag{1}$$

It has been shown by Tuck and Vanden-Broeck [10], that only two steady solutions can exist for this problem. They are the stagnation point solution and the cusp solution. The stagnation point solution is thought to occur for small values of the Froude number and is characterised by a stagnation point on the free surface directly above the sink. The cusp solution occurs for larger values of the Froude number and is characterised by the cusp-like shape of the free surface. In this case, the free surface attaches smoothly to the wall, maintaining a finite, non-zero velocity at the point of contact. It is possible that this type of solution is the forerunner to a solution in which both air and water are drawn directly into the sink, and the interface enters at an angle between 0 and 90 degrees [11]. Numerous examples of both stagnation point solutions [12–17] and cusped solutions [1–3, 10, 16, 18, 19] can be found in the literature.

Most of the work carried out with this particular geometry has been at the extremes of the range γ, that is, $\gamma = 0$ and $\gamma = 1/2$.

In this case where $\gamma = 0$, Tuck and Vanden-Broeck [10] have shown that the cusp solution exists for a unique value of the Froude number (approximately $F = 3.55$). Hocking and Forbes [13] verified this result and showed that steady stagnation point solutions exist for a range of Froude numbers from 0 to about 1.42.

In the finite depth case, $\gamma = 1/2$, most researchers have worked with a Froude number defined slightly differently. Since their region of interest was a fixed finite depth, they found that a more natural definition of the Froude number was one based on the level of the fluid above the lower boundary a long way from the sink. Using this definition for the Froude number, they found that the cusp solution exists in the range $(\sim 1, \infty)$ [16, 19]. Forbes and Hocking [12], and Hocking and Forbes [14] also calculated stagnation point solutions for Froude numbers in the range $[0, \sim 0.3)$.

Little work has been carried out for arbitrary γ. At $\gamma = 1/3$, Sautreaux [2] (see also [1, 3]) discovered that an exact solution with a cusp can be calculated. Vanden-Broeck and Keller [16] and Hocking [18] have numerically calculated cusp solutions for arbitrary values of γ. They found that for each value of γ, except at $\gamma = 1/2$, the cusp solution exists at a unique value of the Froude number.

In the present work, we calculate the boundary of the region in parameter space in which stagnation point solutions occur for arbitrary γ. We compare these with work done at the extremes of the range of γ. When $\gamma = 1/2$, the problem changes (the flow at infinity is no longer zero) and our solution technique is not designed to handle this. To compare our results with work for the case of finite depth, we calculate the solutions as γ approaches $1/2$, and take this to be the limiting behaviour of the fluid at $\gamma = 1/2$.

We find that our solutions are consistent with work done at the extremes of the range of γ, in that for each γ the solution technique converges for a finite range of Froude numbers $0 < F < F_{crit}$. It is still unclear why the solution method fails to converge for Froude numbers greater than the critical value. However, this breakdown is consistent with similar results in related problems, in which integral equation techniques also failed at the same value of F as the series truncation method (see for example [12]).

2. Problem formulation

The solution is obtained using a series truncation method similar to that first used by Tuck and Vanden-Broeck [10]. The method is to define a transformation which maps the region of interest into the lower unit semi-circle. Specifically the transformation maps the free surface of the fluid to the boundary of the semi-circle, and the solid boundary to the real axis. An infinite series with unknown real coefficients is included in the transformation to enable us to satisfy the boundary conditions along the free surface.

The fluid is assumed to be inviscid and incompressible, and the flow to be irrotational, and therefore the equations governing the flow are

$$\nabla^2\Phi(x, y) = 0 \tag{2}$$

subject to the boundary conditions,

$$N_X\Phi_X - \Phi_Y = 0 \tag{3}$$

$$gN(X) + \tfrac{1}{2}(\Phi_X^2 + \Phi_Y^2) = 0\,, \tag{4}$$

on $Y = N(X)$, where $Y = N(X)$ is the equation of the free surface, Φ is the velocity potential, and g is the acceleration due to gravity. In the case of a two-layer fluid with a stagnant upper layer, g would be replaced by $g' = (\Delta\rho/\rho)g$ in this and all subsequent equations. Equation (3) is the kinematic boundary condition, and equation (4) is Bernoulli's equation which ensures that the pressure is constant along the free surface. As $X \to \infty$ the velocity of the fluid tends to zero, so the free surface at infinity has elevation $Y = 0$.

Following the work of Hocking [18], we scale all lengths by $(m^2(1-\gamma)^2/8\pi^2 g)^{1/3}$ and all velocities by $(mg(1-\gamma)/\pi)^{1/3}$ to get

$$\nabla^2\phi = 0 \tag{5}$$

$$\eta_x\phi_x - \phi_y = 0 \tag{6}$$

$$\eta + (\phi_x^2 + \phi_y^2) = 0\,, \tag{7}$$

where $y = \eta(x)$ is the free surface and $\phi(x, y)$ the velocity potential in non-dimensional variables.

The Froude number is given by equation (1), so after scaling, the Froude number in terms of the non-dimensional sink depth, h_s, becomes

$$F = \left(\frac{8\pi^2}{h_s^3}\right)^{1/2} . \tag{8}$$

Following Tuck and Vanden-Broeck [10] and Hocking and Forbes [13], we define a new complex variable t so that

$$e^f = \frac{4t}{(t+1)^2} , \tag{9}$$

where $f = \phi + i\psi$ is the complex velocity potential. This maps the unit circle in the t-plane to an infinite strip in the f-plane (see Fig. 2).

We also transform the region in the z-plane into the lower half circle of the t-plane (see Fig. 2). An appropriate transformation is

$$\frac{dz}{dt} = it^{-\gamma}(t+1)^{2\gamma-2} \sum_{j=0}^{\infty} a_j t^j . \tag{10}$$

Consider now equations (5)–(7), the governing equations for this system.

Equation (5) is satisfied by our choice of transformation (10), since $f(z(t))$ is an analytic function. Equation (6) asserts that the free surface of the fluid is a streamline. Equation (9) ensures that $\psi = 0$ on $t = e^{-i\theta}$, $0 \leq \theta < \pi$. Thus, equation (6) is automatically satisfied by our choice of transformations. The vertical wall above the sink corresponds to $0 < t < 1$, and the

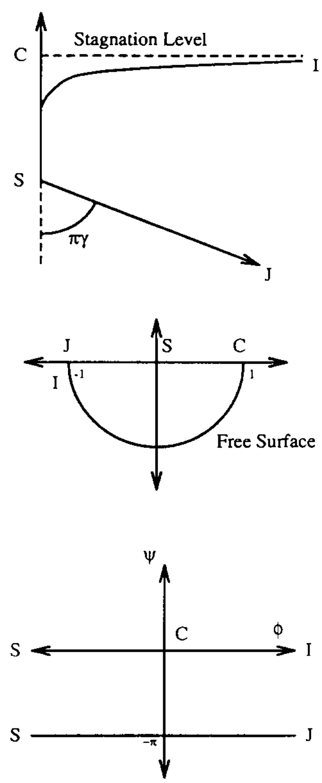

Fig. 2. The geometry of the problem in the z-plane, the t-plane and the f-plane, respectively.

sloping wall beneath the sink to $-1<t<0$, provided all the a_j, $j=0, 1, 2, \ldots$ are real. Note that $f(z) \to (1/1-\gamma)\ln(z+ih_s)$ as $t \to 0$ and $f(z) \to (2/1-2\gamma)\ln z$ as $t \to -1$, i.e. $z \to \infty$, reflecting the restriction of the fluid to the region beneath the line $y=0$. Hence it remains to solve equation (7) on the free surface of the fluid.

Since $\eta(x)$ is the equation of the free surface, it can be written as $\eta(x) = \Im m(z(t(\theta)))$ for $t = e^{-i\theta}$, $\theta \in [0, \pi]$. Making this substitution and using the transformations (9) and (10), equation (7) can be written as,

$$P(\theta, \gamma; a_j) = y(0) + Y(\theta) + \frac{(2\cos(\frac{1}{2}\theta))^{4-4\gamma}\tan^2(\frac{1}{2}\theta)}{A^2(\theta) + B^2(\theta)} = 0\,, \tag{11}$$

where

$$A(\theta) = \sum_{j=0}^{\infty} a_j \cos(j\theta)$$

$$B(\theta) = \sum_{j=0}^{\infty} a_j \sin(j\theta)$$

$$Y(\theta) = -\int_0^{\theta} \left\{ \frac{\sum_{j=0}^{\infty} a_j \sin(js)}{(2\cos(\frac{1}{2}s))^{2-2\gamma}} \right\} \mathrm{d}s\,.$$

If there is a stagnation point on the free surface above the sink, then we require $y(0)=0$.

If the cusp solution is to exist we require that the velocity of the fluid is not zero at $t=1$, the point on the free surface directly above the sink. Thus, we require

$$\sum_{j=0}^{\infty} a_j = 0\,. \tag{12}$$

Equation (11) then gives the location of the cusp point as

$$y(\theta)|_{\theta=0} = -|f'(z(\theta))|^2_{\theta=0} = \frac{2^{2-4\gamma}}{\left(\sum_{j=0}^{\infty} j a_j\right)^2}\,. \tag{13}$$

It is also necessary to compute the sink depth h_s to find the value of F. Since $t=0$ corresponds to the sink, and $t=1$ to the point on the free surface above the sink, the sink depth is

$$h_s = y(t)|_{t=1} - \int_0^1 t^{-\gamma}(1+t)^{2\gamma-2} \sum_{j=0}^{\infty} a_j t^j \,\mathrm{d}t\,. \tag{14}$$

In the case of a cusp solution $y(t)|_{t=1}$ is given by (13), and for stagnation point solutions, $y(t)|_{t=1} = 0$.

3. An exact solution

It was initially shown by Sautreaux [2] (see also Craya [1], Tuck [3]) that there is an exact solution with a cusp when $\gamma = 1/3$. To explore this possibility we set $\gamma = 1/3$ in (11) with (13) and (12) and assume $a_j = 0$, $j = 2, 3, \ldots$. Equation (12) gives $a_0 = -a_1$ and substituting into (11) and solving for a_0 gives $a_0 = (2/3)^{1/3} = -a_1$. So for this exact solution, equation (10) becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -i\left(\frac{2}{3}\right)^{1/3} \frac{1-t}{t^{1/3}(1+t)^{4/3}} . \tag{15}$$

We can integrate (15) to obtain the free-surface shape and the Froude number. From equation (8) we have

$$F = \left(\frac{8\pi^2}{h_s^3}\right)^{1/3} \approx 2.01 ,$$

since $h_s = \frac{1}{2}3^{2/3}\mathbf{B}(\frac{2}{3}, \frac{1}{2}) \approx 2.68$, where $\mathbf{B}$ is the Beta-function [20]. Numerical work by Hocking [18] and Vanden-Broeck and Keller [16] indicates that no steady solutions exist for F slightly larger or smaller than this value.

4. Numerical solution

We cannot solve equations (11), (12) and (13) for general values of γ unless we use numerical techniques. If we truncate the series to N terms and then evaluate (11) at N different values of $\theta_i = (i - \frac{1}{2})\pi/N$, $i = 1, \ldots, N$, we are left with a set of N non-linear algebraic equations, which we can solve using a Newton iteration scheme. Initially, a starting guess for the series coefficients was chosen as the exact solution (for $\gamma = 1/3$) for all cusp simulations, and all zeros for stagnation point simulations at very low values of F. Once these simulations had converged, however, these solutions were used as a starting point for simulations with slightly different parameter values.

Using this scheme, the cusp solutions were computed for a range of values of γ and compared with the work of Hocking [18] and Vanden-Broeck and Keller [16] with agreement to graphical accuracy. At $\gamma = 1/3$, the numerical scheme computed the exact solution accurate to 5 decimal places with $N = 40$. These results show the method is working correctly. Figure 3 shows an example of a cusp solution for the case $\gamma = 1/3$, compared to the stagnation point solution with $F = 0.9$, close to the limit of F for which solutions could be computed.

To compute stagnation point solutions, it is necessary to impose an extra equation on the system to fix the Froude number, and consequently an extra coefficient in the series must be computed. The extra equation is obtained by enforcing equation (14) numerically, thus fixing h_s and therefore F.

Figure 4 shows the shape of the free surface for 3 different values of γ for a fixed value of $F = 0.5$. All are scaled so that the sink is located at $y = -1$. The trough near the stagnation point deepens significantly as γ increases, and the free surface asymptotes to the stagnation level more slowly. In the limit as $\gamma \rightarrow 0.5$, the free surface never returns to the stagnation level, but levels out at a height dictated by the (now non-zero) downstream velocity.

The stagnation point solutions were usually computed to 100 terms, which gave answers accurate to 3 decimal places. Table 1 shows convergence of the series coefficients for increasing values of N for a typical case ($\gamma = 0.25$, $F = 0.8$). Generally, the higher

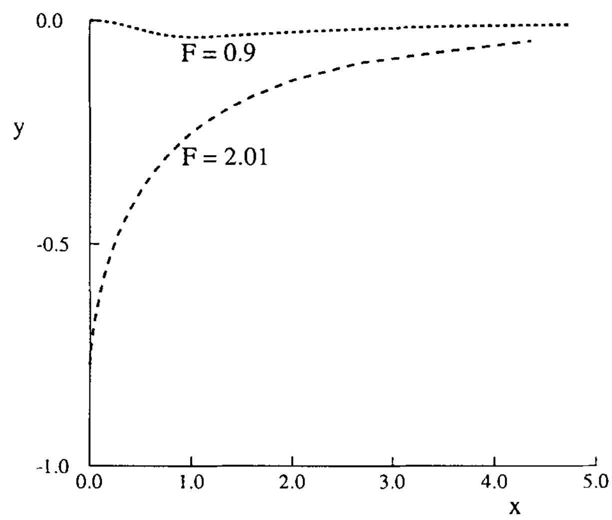

Fig. 3. Comparison of the free surface shape for a stagnation point solution near F_{crit}, $F = 0.9$ with the cusp solution ($F = 2.01$) for the case in which $\gamma = 1/3$. The diagram is scaled so that the sink is located at $y = -1$.

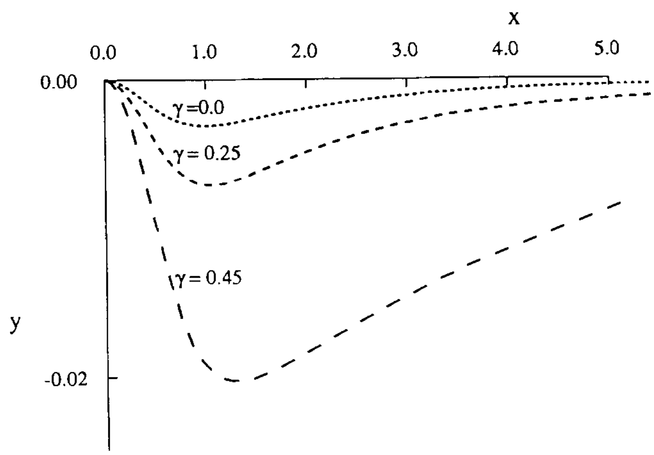

Fig. 4. Comparison of the free surface shape for stagnation point solutions for $F = 0.5$ at values of $\gamma = 0.0$, 0.25 and 0.45. The diagram is scaled so that the sink is located at $y = -1$.

Table 1. Table of the computed series coefficients for $\gamma = 0.25$, $F = 0.8$

N	a_0	a_5	a_{10}	a_{15}	a_{30}
12	5.82411	0.000952	-0.99×10^{-4}	–	–
24	5.82411	0.000983	-1.86×10^{-4}	7.6×10^{-5}	–
48	5.82411	0.000985	-1.92×10^{-4}	8.5×10^{-5}	-1.1×10^{-5}
96	5.82411	0.000985	-1.92×10^{-4}	8.6×10^{-5}	-1.2×10^{-5}
192	5.82411	0.000985	-1.92×10^{-4}	8.6×10^{-5}	-0.9×10^{-5}

coefficients converged more slowly, due to the effects of the truncation. The stagnation point solutions were found to exist for a range of Froude numbers, $0 \leq F < F_{crit}$ for some finite upper bound F_{crit}. The value of F_{crit} was calculated approximately for a series of values of γ, by fixing γ and increasing the Froude number until the algorithm failed to converge. The results obtained are summarised in Fig. 5, which shows the maximum Froude number at

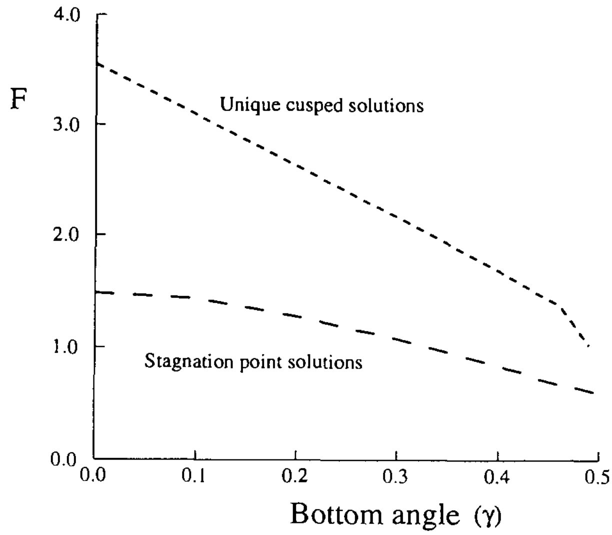

Fig. 5. Plot of regions in parameter space within which solutions have been computed. The cusp solution is unique for any value of γ, but the stagnation point solutions were computed for all values of F beneath the curve shown.

which stagnation point solutions were computed, and the unique Froude number at which cusp solutions exist, for each value of γ in the range.

This behaviour is consistent with the work of Hocking and Forbes [13], and Forbes and Hocking [12] at the extremes of the range of γ. As the Froude number approached F_{crit}, waves began to appear on the free surface, but these were clearly of numerical origin, with the wavelength dependent on the number of collocation points. Forbes and Hocking [3] recently used a formulation for the finite depth case which should find waves on the free surface at small Froude numbers if they exist, but none were found. Thus, the formation of these numerical waves as F approaches F_{crit} probably foreshadows the failure of the numerical scheme. The reason for the failure of the numerical scheme is unclear, and may be related either to the physics or to the method itself. However, the fact that this failure occurs in related problems even using an integral equation approach, (see [12, 14]), suggests that it is not the choice of numerical scheme. Hocking and Forbes [13] calculated the stagnation point solution for the case $\gamma = 0$. They found that the limit of their solutions (for $\gamma = 0$) was at about $F = 1.42$. As can be seen from Fig. 4, our solutions broke down at about $F = 1.45$ (for $\gamma = 0$).

Forbes and Hocking [12], and Hocking and Forbes [14], calculated stagnation point solutions caused by a line sink in a fluid of finite depth for different heights of the sink above the lower boundary. The solution which they obtained when the sink was on the lower boundary corresponds to the limit of the solutions we have calculated as $\gamma \to 0.5$. In the finite depth case, the velocity at infinity is no longer zero, so the free surface compensates by dropping by an amount $(\pi/h_b)^2$, where h_b is the height of the fluid above the lower boundary a long way from the sink [18]. Thus, the sink depth beneath the stagnation point can be expressed as a function of the down-stream depth,

$$h_s = h_b + \left(\frac{\pi}{h_b}\right)^2 .$$

Working with the down-stream Froude number

$$F_b = \left(\frac{2\pi^2}{h_b^3}\right)^{1/2},$$

the limit of our solutions as $\gamma \to 0.5$ was $F_b = 0.31$. This value compares well with that obtained by Forbes and Hocking [12], who found $F_b = 0.3$. Hocking and Forbes [14] were able to compute solutions only up to $F = 0.24$, but there are signs of numerical instability in their method which are independent of the physics.

5. Summary

Following the work of Tuck and Vanden-Broeck [10], and Hocking [18], we have modelled a steady-state withdrawal situation in a water body using a series truncation method. Both types of solutions shown to exist by Tuck and Vanden-Broeck [10] were obtained.

The solutions containing a cusp were in good agreement with earlier work. In the case of flows with a stagnation point, the results were found to be consistent with other work done at the extremes of the range of γ in that for each γ, solutions exist over a range of values of the Froude number, $0 < F < F_{\text{crit}}$, where F_{crit} is some finite upper bound. The values of F_{crit} calculated ranged from $F = 1.48$ at $\gamma = 0$ to $F = 0.575$ at $\gamma = 0.499$. This upper bound in Froude number is consistent with other work on closely related problems, which have been solved using both integral equation and series truncation methods. This suggests that some physical mechanism may be responsible for the breakdown.

For each value of γ, no steady solutions of any kind were found above this critical value of F, until a second critical value of F was reached at which the cusp solution occurred. Above this second critical value, no further steady solutions were obtained.

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